New Explicit Expressions for the Coupling Matrix Elements Related to Scattering from a Planar Periodic Single-Strip Grating

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Abstract— **This paper presents an alternative approach for finding elements of the coupling matrix in the multimode equivalent network that describes plane-wave scattering from a planar periodic single-strip grating at a dielectric interface. After solving a Fredholm integral equation of the first kind whose kernel has a logarithmic singularity, the coupling matrix elements are obtained in an explicit analytical form, thus avoiding numerical integration. Numerical computations are carried out and the results are compared to some of the existing ones.**

I. Introduction

A RIGOROUS multimode network formulation for the problem of TE or TM plane wave scattering from a planar periodic metal-strip grating at a dielectric interface is proposed in [1]. In that formulation the grating is represented by a mutual coupling matrix which does not depend on angle of incidence and frequency, except for a multiplying constant. The incident wave as well as reflected and transmitted space harmonics are modeled by transmission lines which connect into the mutual coupling matrix. This matrix is in the form of an impedance matrix for the aperture formulation in which the electric field between the strips is chosen as the unknown. Alternatively, in the case of the obstacle formulation, where current on the strip is the unknown, the coupling matrix has the form of an admittance matrix. In addition to these two formulations there are two possible wave polarizations, transverse electric (TE), and transverse magnetic (TM), so that four combinations can be made. In each of these combinations, the coupling matrix elements are different, but are always related to an integral over the grating unit cell of an unknown function which is to be found from an integral equation. This integral equation has one particular form for the TM-obstacle and TE-aperture cases, and a different form for the TM-aperture and TE-obstacle cases. An approximate small-argument solution for each of the four possible cases was given in [2].

A novel rigorous solution of the integral equation for the cases of TM-obstacle and TE-aperture formulations was proposed in [3]. The method of the solution was to reduce the integral equation for these two formulations to a singular Cauchy-type integral equation, which has a known solution. From this, the coupling matrix elements were found in the form of double integrals that have to be evaluated numerically.

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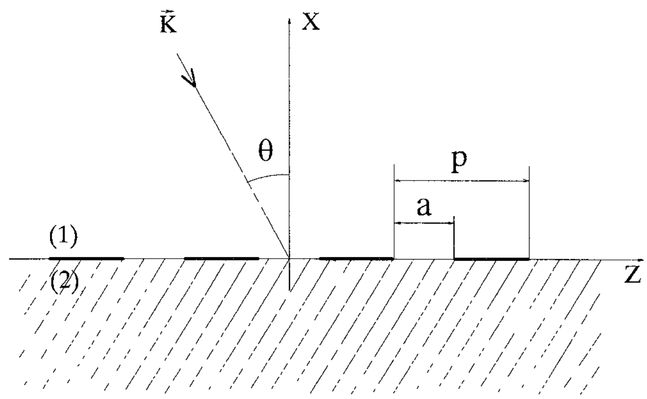

Fig. 1. A planar periodic single-strip metal grating at a dielectric interface. A plane wave is incident at an angle. Both polarizations are considered.

In [4], the method of [3] is extended to the case of a double-strip grating. Although the method is basically the same as for a single-strip grating, it is mathematically more involved.

In this paper we use the multimode network representation derived in [1], for the problem of TE or TM plane wave scattering from a planar periodic single-strip grating at a dielectric interface. However, to find the coupling matrix elements the alternative approach is chosen, i.e. the starting integral equation is the one for TE-obstacle and TM-aperture formulations. It is a Fredholm-type equation of the first kind, whose kernel has a logarithmic singularity. It is known [5] that an equation of this type has a closed-form solution. After solving the integral equation the coupling matrix elements A_{mn} are obtained in a simple analytical form. Therefore, no numerical integration is needed, as opposed to [3].

To check the validity of this approach some computations have been carried out and the results are compared to those of [3].

II. The Equivalent Network Representation and the Corresponding Integral Equation

Geometry of the problem under consideration is shown in Fig. 1. A plane wave is incident at an angle θ upon a planar periodic zero-thickness single-strip grating at a dielectric interface. The grating has period p, the distance between the strips is a, and the parameters of media (1) and (2) are $\epsilon_r^{(1)}, \mu_r^{(1)}$ and $\epsilon_r^{(2)}, \mu_r^{(2)}$ respectively. The polarization can be either TE or TM to x. The rigorous equivalent networks, derived in [1] for the cases of TE-obstacle and TM-aperture formulations are shown in Figs. 2 and 3 respectively. The quantities in these

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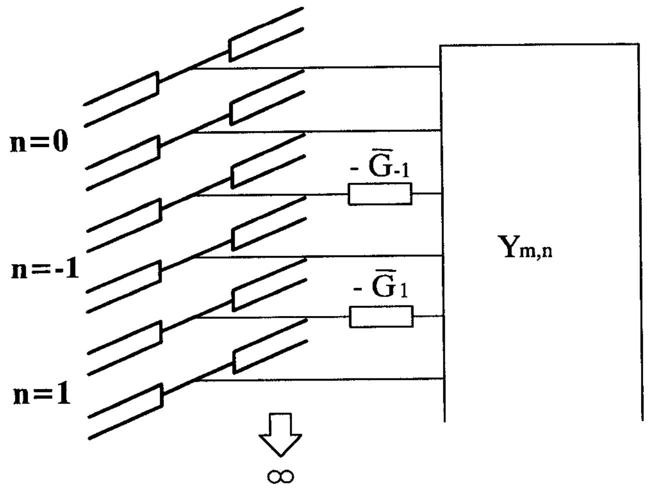

Fig. 2. The rigorous equivalent network for the case of TE-obstacle formulation.

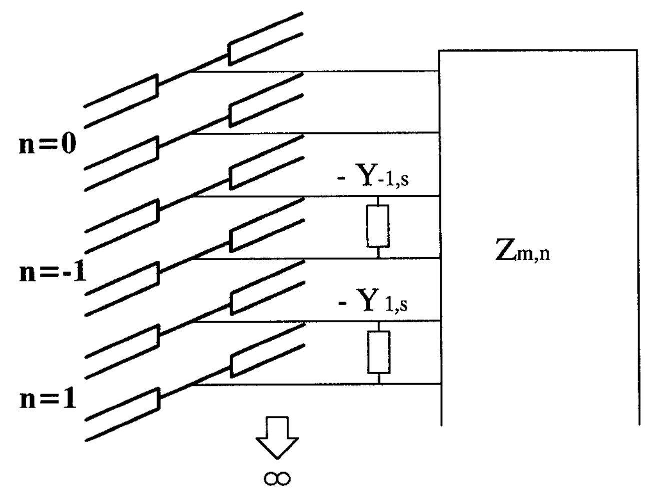

Fig. 3. The rigorous equivalent network for the case of TM-aperture formulation.

figures are [1]

$$\bar{G}_n = \frac{j\omega\mu_0 p}{2\pi|n|}\frac{1}{\frac{1}{\mu_r^{(1)}}+\frac{1}{\mu_r^{(2)}}}, \quad Y_{n,s} = \frac{j\omega\epsilon_0 p}{2\pi|n|}\left(\epsilon_r^{(1)}+\epsilon_r^{(2)}\right).$$

The coupling matrix elements A_{mn} (representing admittances Y_{mn} in the case of TE-obstacle formulation, and impedances Z_{mn} in the case of TM-aperture formulation) are to be found from

$$A_{m,n} = \int_{-b/2}^{+b/2} f_n(z) e^{j\frac{2m\pi}{p}z}\, dz. \tag{1}$$

The unknown function $f_n(z)$ which appears in (1) satisfies the following integral equation

$$\int_{-b/2}^{+b/2} f_n(z')\sum_{m=0}^{\infty} B\frac{e^{-j\frac{2m\pi}{p}(z-z')}}{|m|}\, dz' = e^{-j\frac{2n\pi}{p}z} \tag{2}$$

where

$$B = \begin{cases} \frac{j\omega\mu_0 p}{2\pi}\frac{1}{\frac{1}{\mu_r^{(1)}}+\frac{1}{\mu_r^{(2)}}} & \text{for TE-obstacle formulation} \\ \frac{j\omega\epsilon_0 p}{2\pi}\left(\epsilon_r^{(1)}+\epsilon_r^{(2)}\right) & \text{for TM-aperture formulation} \end{cases}$$

In (1) and (2) $b = a$ for the TM-aperture formulation (the origin is in the center of the aperture), and $b = p - a$ in the case of TE-obstacle formulation (the origin in Fig. 1 is then shifted to the center of the strip).

III. Solution of the Integral Equation and the Explicit Expressions for the Coupling Matrix Elements

By using the formula

$$\sum_{m=1}^{\infty}\frac{\cos(mz)}{m} = \ln\frac{1}{2|\sin(z/2)|} = \ln\frac{1}{2\sin(|z|/2)}$$

which is valid for $0 < z < 2\pi$, (2) can be rewritten as

$$2B\int_{-b/2}^{+b/2} f_n(z')\ln\frac{1}{\sin\frac{\pi|z-z'|}{2}}\, dz' = e^{-j\frac{2n\pi}{p}z}. \tag{3}$$

A change of variables

$$\frac{\pi}{p}z' = \frac{\xi'}{2}, \qquad \frac{\pi}{p}z = \frac{\xi}{2}$$

puts (3) into the following form

$$\int_{-A}^{+A} F_n(\xi')\ln\frac{1}{\sin\frac{|\xi-\xi'|}{2}}\, d\xi' = \varphi_n(\xi) \tag{4}$$

where

$$F_n(\xi) = \frac{Bp}{\pi}f_n\left(\frac{p\xi}{2\pi}\right) \tag{5}$$

$$\varphi_n(\xi) = e^{-jn\xi} \tag{6}$$

$$A = \frac{\pi b}{p}. \tag{7}$$

Equation (4) is a Fredholm-type integral equation of the first kind whose kernel has a logarithmic singularity. It is known [5] that an equation of this type has a closed-form solution

$$\begin{aligned} F_n(\xi) = &-\frac{1}{2\pi^2}\frac{1}{\ln\sin(A/2)}\frac{\cos(\xi/2)}{\sqrt{\cos\xi-\cos A}} \\ &\times\int_{-A}^{A}\varphi_n(\xi')\frac{\cos(\xi'/2)}{\sqrt{\cos\xi'-\cos A}}\, d\xi' \\ &-\frac{1}{2\pi^2}\frac{1}{\sqrt{\cos\xi-\cos A}}\text{v.p.} \\ &\times\int_{-A}^{A}\varphi_n'(\xi')\frac{\sqrt{\cos\xi''\cos A}}{\sin\frac{\xi'-\xi}{2}}\, d\xi' \end{aligned} \tag{8}$$

where the symbol v.p. stands for the principal value.

The substitution of $\varphi_n(\xi)$ from (6) into (8), and the subsequent evaluation of the integrals (which is given in the appendix) leads to

$$\begin{aligned} F_n(\xi) = &-\frac{1}{2\pi\sqrt{2}}\frac{1}{\ln\sin(A/2)}\frac{\cos(\xi/2)}{\sqrt{\cos\xi-\cos A}} \\ &\times(P_n(u)+P_{n-1}(u)) + \frac{n}{\pi\sqrt{2}}\frac{1}{\sqrt{\cos\xi-\cos A}} \\ &\times\begin{cases} -\sum_{p=0}^{-n}\rho_{-n-p}(u)e^{j\xi(p+1/2)} & n\le -1 \\ \sum_{p=0}^{n}\rho_{n-p}(u)e^{-j\xi(p+1/2)} & n\ge 1 \end{cases}. \end{aligned} \tag{9}$$

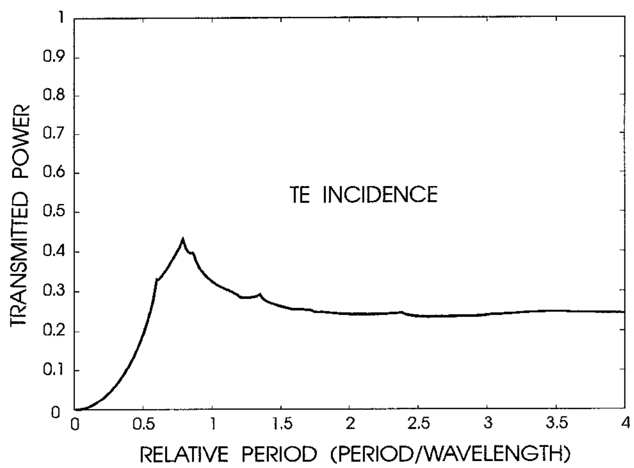

Fig. 4. The transmitted power in the lowest (n =0) mode versus the relative period for the case of TE-incidence. The parameters are taken to be: $a/p = 0.5, \theta = 15°$, $\epsilon_r^{(2)} = 2$ and $\epsilon_r^{(1)} = \mu_r^{(1)} = \mu_r^{(2)} = 1$. The network used is the one shown in Fig. 2.

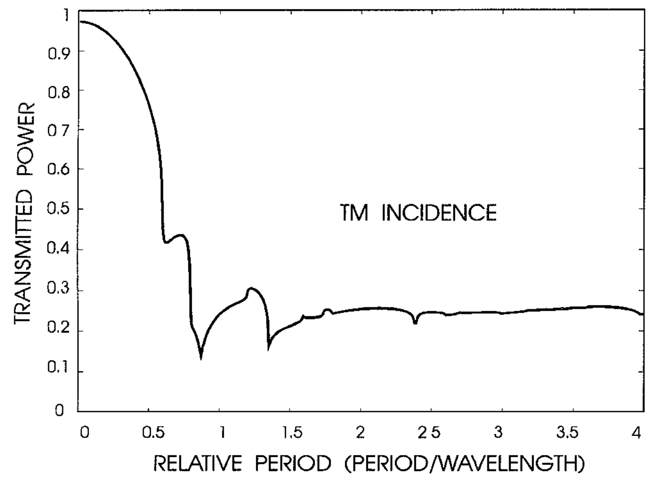

Fig. 5. Same as Fig. 4 but for TM-incidence. The network used is the one from Fig. 3.

Here, P_k are the Legendre polynomials, and ρ_k are related to the Legendre polynomials and are given in the appendix. The argument of P_k and ρ_k is $u = \cos A$. Note that the second integral in (8) is equal to zero for $n = 0$.

Now, the coupling matrix elements can be obtained as follows: First, make a change of variables in (1) and (2)

$$\pi z/p = \xi$$

Then, taking into account (5) and (7)

$$A_{m,n} = \frac{1}{2B}\int_{-A}^{A} F_n(\xi)e^{jm\xi}\,d\xi$$

Substituting here the function $F_n(\xi)$ from (9), and using the Dirichlet formula [6] for the Legendre polynomials

$$\begin{aligned} P_n(\cos\theta) &= \frac{\sqrt{2}}{2\pi}\int_{-\theta}^{\theta}\frac{e^{j(n+1/2)\varphi}\,d\varphi}{\sqrt{\cos\varphi-\cos\theta}} \\ &= \frac{\sqrt{2}}{\pi}\int_{0}^{\theta}\frac{\cos(n+1/2)\varphi\,d\varphi}{\sqrt{\cos\varphi-\cos\theta}} \end{aligned} \tag{10}$$

and the relation [6]

$$P_k(u) = P_{-k-1}(u)$$

one obtains the final expression for the coefficients $A_{m,n}$

$$A_{m,n} = \frac{1}{2B}\left[\frac{(P_n(u)+P_{n-1}(u))(P_m(u)+P_{m-1}(u))}{4\ln\sin(A/2)} + n\begin{cases} -\sum\limits_{p=0}^{-n}\rho_{-n-p}(u)P_{m+p}(u) & n \le -1 \\ \sum\limits_{p=0}^{n}\rho_{n-p}(u)P_{-m+p}(u) & n \ge 1 \end{cases}\right]. \tag{11}$$

Thus, the coupling matrix elements are found in a simple explicit analytical form.

IV. Numerical Results

In order to check the validity of this approach to plane wave scattering from a planar periodic single-strip grating at a dielectric interface, some numerical computations have been carried out.

The transmitted power in the lowest ($n = 0$) mode versus the relative grating period p/λ_0, for the case of TE-incidence is shown in Fig. 4. The relevant network is the one in Fig. 2, and the parameters from Fig. 1 have the following values: $\theta = 15°, a/p = 0.5, \epsilon_r^{(2)} = 2$ and $\epsilon_r^{(1)} = \mu_r^{(1)} = \mu_r^{(2)} = 1$.

Fig. 5 shows the transmitted power in the lowest mode versus the relative period in the case of TM-incidence,with the same parameter values. The equivalent network used is shown in Fig. 3.

In these computations modes of order up to ± 6 have been included. As expected, the results are identical to those in [3]. The corresponding curves are indistinguishable.

V. Conclusion

Starting from the multimode network representations for the problem of plane wave scattering from a planar periodic single-strip zero-thickness metal grating at a dielectric interface, new simple explicit expressions for the coupling matrix elements have been obtained. Therefore, no numerical integration is needed to obtain values of these elements. The relevant integral equation to solve was a Fredholm-type equation whose kernel has a logarithmic singularity. A closed-form solution for this equation is presented.

Appendix

In this appendix we evaluate the two integrals appearing in (8). The first one is given by

$$\int_{-A}^{A}\frac{e^{-jn\xi'}\cos(\xi'/2)\,d\xi'}{\sqrt{\cos\xi'-\cos A}} = \frac{n}{\sqrt{2}}(P_n(\cos A)+P_{n-1}(\cos A)). \tag{A1}$$

In deriving (A1) the Dirichlet formula (10) and the trigonometric identity $\cos a \cos b = (1/2)(\cos(a+b)+\cos(a-b))$ were used.

The other integral in (8) which is understood in the sense of principal value can be evaluated by means of residues from complex function theory.

First, the sine and cosine terms under the integral sign are transformed by using Euler's formula and a change of variables

$$e^{j\xi'} = z, \qquad e^{j\xi} = z_0 \tag{A2}$$

is made to give

$$\text{v.p.} \int_{-A}^{A} \frac{-jne^{-jn\xi'}\sqrt{\cos\xi' - \cos A}}{\sin\frac{\xi'-\xi}{2}}\, d\xi' = -jn\sqrt{2}\sqrt{z_0}\,\text{v.p.} \int_{\Gamma} \frac{\sqrt{(z-\alpha)(z-\alpha^*)}}{z^{n+1}(z-z_0)}\, dz \tag{A3}$$

where $\alpha = e^{jA}$ and $*$ implies complex conjugate value. The contour of integration Γ is the arc of the unit circle $|z| = 1$ between the points $z = \alpha^*$ and $z = \alpha$. The square root function is a two-valued function with two branch points $z = \alpha$ and $z = \alpha^*$. By making a cut along the unit circle between the branch points this function becomes analytical and single-valued, the branch being specified by the condition $w(0) = +1$. The contour of integration Γ goes along the inner lip of the cut. Let Γ' be the closed contour going along the inner and outer lips of the cut in the clockwise direction. Then, the integral along Γ is equal to one half of the integral along Γ', since the square root changes sign on the outer lip of the cut. But, the integral along the closed contour Γ' can be evaluated by means of the residues at the two singular points $z = 0$ and $z = \infty$. Therefore, (A.3) can be written as

$$\text{v.p.} \int_{-A}^{A} \frac{-jne^{-jn\xi'}\sqrt{\cos\xi' - \cos A}}{\sin\frac{\xi'-\xi}{2}}\, d\xi' = -jn\sqrt{2}\sqrt{z_0}\frac{1}{2}2\pi j(\text{Res}f(0) + \text{Res}f(\infty)) \tag{A4}$$

where $f(z)$ is the function under the integral sign on the right-hand side of (A3).

To find the residues at $z = 0$ and $z = \infty$ the function $f(z)$ has to be developed into the Laurent series in the vicinity of these two points, i.e. for $|z| < |z_0| = 1$ and for $|z| > |z_0| = 1$.

First, note that the function $[(z-\alpha)(z-\alpha^*)]^{-1/2}$ where $\alpha = e^{jA}$ is the generating function for the Legendre polynomials [6]

$$[(z-\alpha)(z-\alpha^*)]^{-1/2} = \sum_{n=0}^{\infty} P_n(\cos A) z^n, \quad |z| < 1.$$

Inverting the power series in the last relation gives

$$[(z-\alpha)(z-\alpha^*)]^{1/2} = \sum_{n=0}^{\infty} \rho_n(\cos A) z^n, \quad |z| < 1 \tag{A5}$$

where

$$\rho_0(\cos A) = 1, \rho_1(\cos A) = -\cos A$$
$$\rho_n(\cos A) = P_n(\cos A) - 2\cos A P_{n-1}(\cos A) + P_{n-2}(\cos A), \; n \geq 2.$$

In the case $|z| > 1$ one may replace z in (A5) by $1/z$. The result is

$$[(z-\alpha)(z-\alpha^*)]^{1/2} = -z\sum_{n=0}^{\infty} \rho_n(\cos A) z^{-n}, \quad |z| > 1. \tag{A6}$$

The minus sign comes from the fact that the square root changes sign across the cut. Also

$$(z-z_0)^{-1} = \begin{cases} -\sum\limits_{p=0}^{\infty} z_0^{-p-1} z^p, & |z| < 1 \\ \sum\limits_{p=0}^{\infty} z_0^p z^{-p-1}, & |z| > 1 \end{cases}. \tag{A7}$$

Now the Laurent expansions of the function $f(z)$ follow from (A5), (A6) and (A7)

$$f(z) \equiv \frac{\sqrt{(z-\alpha)(z-\alpha^*)}}{z^{n+1}(z-z_0)} = \begin{cases} -\frac{1}{z^{n+1}} \sum\limits_{p=0}^{\infty} z_0^{-p-1} z^p \sum\limits_{k=0}^{\infty} \rho_k(\cos A) z^k, & |z| < 1 \\ -\frac{1}{z^{n+1}} \sum\limits_{p=0}^{\infty} z_0^p z^{-p-1} \sum\limits_{k=0}^{\infty} \rho_k(\cos A) z^{-k}, & |z| > 1 \end{cases}. \tag{A8}$$

The residue of the function $f(z)$ at $z = 0$ is the coefficient of z^{-1} in the Laurent series (A8) for $|z| < 1$. It is

$$\text{Res}f(0) = \begin{cases} 0, & n \leq -1 \\ -\sum\limits_{p=0}^{n} \rho_{n-p}(\cos A) z_0^{-p-1}, & n \geq 0 \end{cases}. \tag{A9}$$

The residue of $f(z)$ at the point $z = \infty$ is the coefficient of z^{-1} in the Laurent series (A.8) for $|z| > 1$ with the opposite sign

$$\text{Res}f(\infty) = \begin{cases} \sum\limits_{p=0}^{-n} \rho_{-n-p}(\cos A) z_0^p, & n \leq -1 \\ 1, & n = 0 \\ 0, & n > 0 \end{cases}. \tag{A10}$$

Finally, (9) is obtained from (8), (A1), (A2), (A4), (A9) and (A10).

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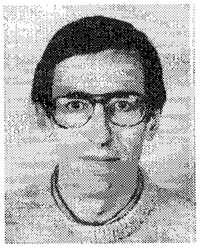

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